Comment on: Role of Intermittency in Urban Development: A Model of Large-Scale City Formation

In Ref. [1] a model for large-scale city formation was proposed. It is based on the discreet dynamics (eqs. (1,2) of [1])

$$n_i(t+1) = (1-\alpha)n_i(t)f_i(t) + \frac{\alpha}{4}\sum_{j\ nn\ i} n_j(t)f_j(t), \quad (1)$$

where $n_i(t)$ is the population at site i of a square lattice, and $f_i(t) \geq 0$ are random multiplication factors drawn independently from a distribution $P(f)$ with the property $\langle f \rangle = 1$. The authors claim that such a multiplicative process with diffusion gives rise to a stationary state with a power law distribution of city sizes n_i $P(n) \sim n^{-\tau}$, where $\tau \simeq 2$ in their simulations is in agreement with empirical observations or real city size distribution.

We demonstrate here that *i)* Eq. (1), with $\langle f \rangle = 1$, does not lead to a stationary state; *ii)* the probability distribution of n_i does not have a power-law tail; *iii)* both stationarity and a power-law tails of the distribution are recovered if a constant source term is added to the RHS of the Eq. (1), but then the uniqueness of the exponent $\tau = 2$ is lost.

We have performed a numerical simulation of the dynamical rules (1) and found that, contrary to the claims of Ref. [1], these dynamical rules *do not lead to a stationary state*. Instead, the total population $N(t) = \sum_i n_i(t)$ decays to zero exponentially in time. For example, for $\alpha = 0.25$, and f being equal to 2 or 0 with equal probabilities, we found that after 1000 time steps $N(t)$ is of order 10^{-15} in clear contradiction with the results of Ref. [1] for the same set of parameters. This is not a finite size effect (we found the same rate of decay for sizes in the range 64^2–200^2). This inconsistency suggests that there is some ingredient in the model simulated by Zanette and Manrubia, which was not reported in their paper.

To understand better the reasons for this exponential decay let us consider first the case $\alpha = 0$, where the population $n_i(t)$ at each site undergoes a random multiplicative process $n_i(t+1) = n_i(t)f_i(t)$ and is uncoupled from other sites. For $\langle f \rangle = 1$ the expectation value $\langle n_i(t) \rangle$ does not change in time. It is known, however, that in such process, $\langle n_i(t) \rangle$ is dominated by extremely rare events, when n_i is exponentially large in t. By the virtue of Central Limit Theorem, in any *typical* realization, $n_i(t) \simeq e^{\langle \ln f \rangle t}$. One can show that for any $P(f)$ such that $\langle f \rangle = 1$, one has $\langle \ln f \rangle < 0$, so that the typical $n_i(t)$ vanishes exponentially for $t \to \infty$. The diffusion ($\alpha > 0$) alone cannot reverse this typical decay. Indeed, the total population also undergoes a multiplicative process: $N(t+1) = F(t)N(t)$, where $F = \sum_i f_i n_i / N$ is the "population" average of f_i. Again $\langle F \rangle = 1$ implies that $\langle \ln F \rangle < 0$, so that typically $N(t) \sim e^{\langle \ln F \rangle t} \to 0$ for $t \to \infty$.

Note that Eq. (1) is the equation for the partition function $\sum_i n_i(t)$ of a directed polymer of length t in a three-dimensional random media [2], and $-\langle \ln F \rangle$ is the polymer's free energy per unit length. For this problem it is known [2] that $P(n)$ is not a power law but rather a log-stretched-exponential law.

On the other hand, if a positive constant is added to the right hand side of Eq. (1), it describes a multiplicative noise process with a lower wall. The purpose of this extra source term is to prevent $n_i(t)$ from becoming too small, while for large n_i its influence can be safely neglected. Such rules dynamics have received much attention recently both for $\alpha > 0$ [3], and $\alpha = 0$ [4]. For $\alpha = 0$ it was shown [4] that a stationary state exists, provided $\langle \ln f \rangle < 0$. The distribution of n has a power law tail $n^{-\tau}$ with τ determined by $\langle f^{\tau-1} \rangle = 1$. Clearly $\langle f \rangle = 1$ implies $\tau = 2$, but the condition $\langle f \rangle = 1$ is no longer necessary for the existence of a stationary state, and cannot be justified on these grounds. For $\alpha > 0$ we found that the system is stationary (it is in the "pinned" phase [3]) and $P(n)$ still has a power law tail with $\tau = 2$. This can be justified by the fact that the correlation length $\xi(\alpha)$ is finite away from the depinning transition [3]. Therefore, a large system can be divided into many virtually uncoupled blocks, whose total population $\tilde{n}_x = \sum_{|i-x|<\xi(\alpha)} n_i$ undergoes a multiplicative dynamics $\tilde{n}_x(t+1) = \tilde{f}_x(t)\tilde{n}_x(t)$ with $\langle \tilde{f} \rangle = 1$. The last condition ensures that $\tau = 2$ for any α. If $\langle f \rangle \neq 1$ instead τ depends on α.

In short, the results of ref. [1] are inconsistent with their rules of dynamics (1), but compatible with the behavior of models with a lower wall [3]. For such models a stationary state does not require $\langle f \rangle = 1$, and any power law exponent $\tau > 1$ is feasible.

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